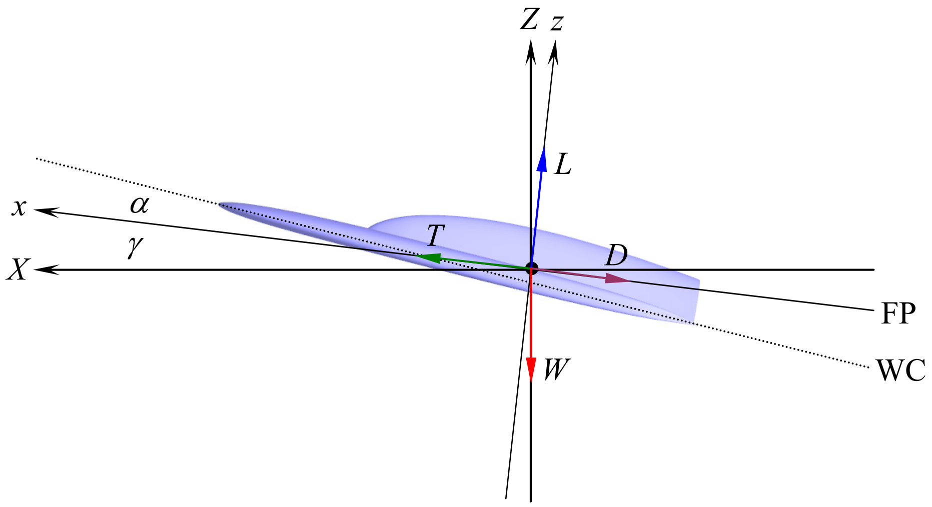
Derivation of Space Plane Equation



L = Lift
D = Drag
T = Thrust
W = Weight
α = Attack Angle
γ = Climb Angle
FP = Flight Path
WC = Wing Chord

Summation of forces along the vertical axis (Z) yields,

$$T\sin\gamma \;-\; D\sin\gamma \;+\; L\cos\gamma \;-\; W \;=\; m\,a\sin\gamma \;-\; m\,\frac{(v\cdot\cos\gamma)^2}{(R_e+h)}$$

For $\gamma \le 5^\circ$, $\sin\gamma \simeq 0$ and $\cos\gamma \simeq 1$. We shall retain the $\cos^2\gamma$ term in the inertial lift I, which is a function of flight speed v and altitude h, while R_e is Earth's radius.

$$L = W - I \qquad \text{where} \qquad I = m\,\frac{(v\cdot\cos\gamma)^2}{(R_e+h)}$$

Using $W = m\,g_h$, we shall define the inertia-to-weight ratio I/W,

$$I/W = \frac{(v\cdot\cos\gamma)^2}{(R_e+h)\,g_h} \qquad \text{where} \qquad g_h = g_e\left(\frac{R_e}{R_e+h}\right)^2 \qquad g_e = 9.807\,\text{m/s}^2 \qquad R_e = 6378\,\text{km}$$

Thus the lift force can be expressed as follows,

$$L = m\,g_h\,(1 - I/W)$$

Introducing the aerodynamic lift-to-drag ratio, L / D, the drag force can now be expressed,

$$D = m\, g_h \, \frac{1 - I / W}{L / D}$$

The differential change in linear momentum, $d\vec{p}$, of the space plane is equal to its differential impulse, $d\vec{J}$, received during a time interval dt,

$$d\vec{p} = d\vec{J}$$

Applying this equation along the flight path direction (x) as shown in the figure on page 4, while using a stationary system analysis,

$$\left(m - dm_{gas}\right)\left(v + dv\right) + dm_{gas}\left(v - u\right) + dm_{air}\left(v - u\right) - mv = -\left(D + W \sin\gamma\right) dt$$

where m and v are the mass and speed of the space plane, dm_{gas} is the differential gas mass and u its relative exhaust speed with respect to the vehicle. We shall assume the vehicle is in rocket mode such that dm_{air} is zero for now. We will analyze the air-breathing mode later.

$$mv + m\,dv - v\,dm_{gas} - dv\,dm_{gas} + v\,dm_{gas} - u\,dm_{gas} - mv \; = \; -\left(D + W \sin\gamma\right) dt$$

Higher order differentials like $dv\,dm_{gas}$ can be neglected, and after simplifying terms,

$$m\,dv - u\,dm_{gas} \; = \; -\left(D + W \sin\gamma\right) dt$$

The sum of the vehicle mass and expelled gas mass is equal to the takeoff mass m_0 at all times,

$$m(t) + m_{gas}(t) = m_0 \qquad \frac{dm}{dt} + \frac{dm_{gas}}{dt} = 0 \qquad dm = -dm_{gas}$$

And after substituting the previously derived relation for the vehicle drag D,

$$m\,dv + u\,dm = -\,m\, g_h \left(\frac{1 - I / W}{L / D} + \sin\gamma \right) dt$$

$$dv = -\,u \frac{dm}{m} \; - \left(\frac{1 - I / W}{L / D} + \sin\gamma \right) g_h \, dt$$

The inertia-to-weight ratio, I / W, is a function of speed v, but since it is impossible to fully separate the variables, we shall assume I / W to be constant over the integration range. Thus,

$$\Delta v = -u \cdot \ln\left(\frac{m_1}{m_0}\right) - \left(\frac{1 - I/W}{L/D} + \sin\gamma\right) \cdot g_h \cdot \Delta t$$

Expressing the exhaust speed u in terms of the specific impulse, $I_{sp} = u / g_e$, and inverting the mass fraction within the natural logarithm, the final result for the space plane equation is,

$$\Delta v = I_{sp} \cdot g_e \cdot \ln\left(\frac{m_0}{m_1}\right) - \left(\frac{1 - I/W}{L/D} + \sin\gamma\right) \cdot g_h \cdot \Delta t$$

$$\frac{m_1}{m_0} = \exp\left[\frac{\Delta v + \left((1 - I/W) \cdot (L/D)^{-1} + \sin\gamma\right) \cdot g_h \cdot \Delta t}{-I_{sp} \cdot g_e}\right]$$

$$g_e = 9.807\,\text{m/s}^2 \qquad g_h = g_e\left(\frac{R_e}{R_e + h}\right)^2 \qquad R_e = 6378\,\text{km}$$

$$I/W = \frac{(v_0 \cdot \cos\gamma)^2}{(R_e + h)\, g_h} \qquad \text{(Inertia-to-Weight Ratio)} \qquad \Delta v = v_1 - v_0$$

Assumptions:
1. Climb Angle $\gamma \le 5^\circ$, $L \simeq W - I$
2. Constant Inertia, $I/W \simeq f(v_0)$
3. Altitude $h \le 105\,\text{km}$, $D > 0$

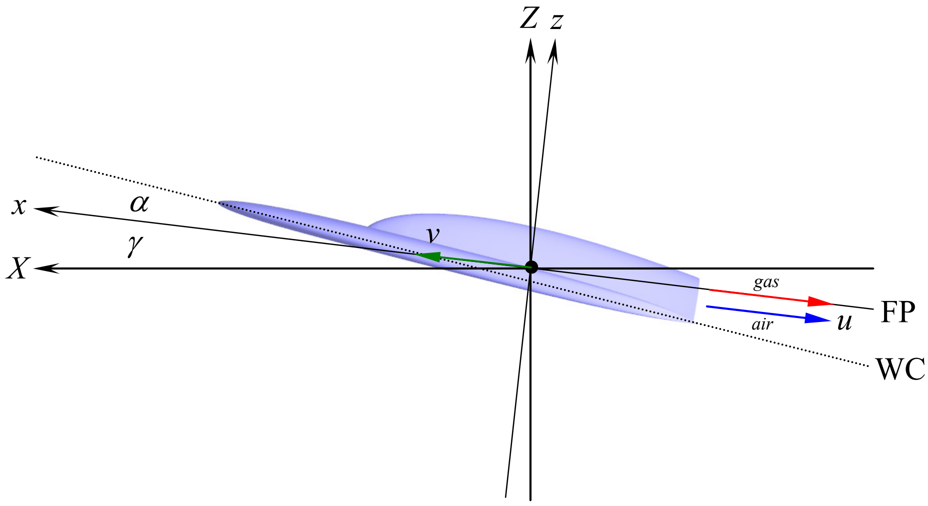
Evaluating the inertia-to-weight ratio, I/W, at the start of the propellant gas burn, by using v_0, will underpredict the speed increase Δv and overpredict the propellant expenditure.

If the space plane is in air-breathing mode, the impulse momentum equation, $d\vec{p} = d\vec{J}$, will retain the additional air term. Applying a stationary system analysis along the flight path (x),

$$(m - dm_{gas})(v + dv) + dm_{gas}(v - u) + dm_{air}(v - u) - mv = -(D + W\sin\gamma)\,dt$$

With respect to the stationary system, which includes the vehicle, its exhaust, and a small portion of the surrounding air, the intake air is initially at rest and has the same exhaust speed as the gas, $(v - u)$. But since the air intake and exhaust masses are equal, they do not directly affect the loss of vehicle mass during the propellant gas burn.

However, the augmented air results in an additional thrust, F_{air}, acting along the positive x direction, based on a moving control volume analysis, with v and u being the intake and exhaust speeds relative to the moving vehicle,



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$$F_{air} = \frac{dm_{air}}{dt}(u - v) \qquad F_{air}\,dt = dm_{air}(u - v)$$

Thus the stationary system analysis and moving control volume analysis yield the same result, not surprisingly, if we were to move $F_{air}\,dt$ to the left side of the equation,

$$(m - dm_{gas})(v + dv) + dm_{gas}(v - u) - mv = -(D + W\sin\gamma - F_{air})\,dt$$

The added thrust from the augmented airflow is reflected in a higher specific impulse compared to the pure rocket mode when the air intake is closed. Therefore, the space plane equation does not need to be modified for air-breathing mode, as long as the specific impulse is accurately determined from the thrust specific fuel consumption $TSFC$, or short SFC,

$$I_{sp} = (SFC \cdot g_e)^{-1} \qquad SFC = \frac{\dot{m}_{fuel}}{T} \qquad \dot{m}_{fuel} = \frac{dm_{fuel}}{dt}$$

In a rocket based combined cycle (RBCC) engine, the fuel flow rate is essentially the gas flow rate, since the embedded rocket exhaust is hydrogen rich when mixed with incoming air. Thus the term "fuel" shall include both liquid hydrogen and liquid oxygen in air-breathing mode.

RBCC: $\dot{m}_{fuel} = \dot{m}_{gas}$

In summary, the space plane equation shown on page 3 is equally valid in air-breathing and pure rocket propulsion mode under the stated assumptions (1-3). It yields an upper bound of propellant gas expenditure when applied to segments of the flight that use an engine mode of nearly constant, or reasonably averaged, specific impulse.