Space Plane Ascent Profile

$$\Delta v = I_{sp} \cdot g_e \cdot \ln\left(\frac{m_0}{m_1}\right) - \left(\frac{1 - I/W}{L/D} + \sin\gamma\right) \cdot g_h \cdot \Delta t$$

$$\frac{m_1}{m_0} = \exp\left[\frac{\Delta v + \left((1 - I/W)\cdot(L/D)^{-1} + \sin\gamma\right)\cdot g_h \cdot \Delta t}{-I_{sp}\cdot g_e}\right]$$

$g_e = 9.807\,\text{m/s}^2$ $\qquad$ $g_h = g_e\left(\frac{R_e}{R_e + h}\right)^2$ $\qquad$ $R_e = 6378\,\text{km}$

$I/W = \frac{(v_0 \cdot \cos\gamma)^2}{(R_e + h)\,g_h}$ $\qquad$ (Inertia-to-Weight Ratio) $\qquad$ $\Delta v = v_1 - v_0$

Assumptions:
1. Climb Angle $\gamma \le 5^\circ$, $L \simeq W - I$
2. Constant Inertia, $I/W \simeq f(v_0)$
3. Altitude $h \le 105\,\text{km}$, $D > 0$

Ejector Burn: $0 < h < 27\,\text{km}$

Acceleration due to Gravity at 27 km: $g_h = 9.724\,\text{m/s}^2$

$$g_h = g_e\left(\frac{R_e}{R_e + h}\right)^2$$

$$g_h = 9.807\,\text{m/s}^2\left(\frac{6378\,\text{km}}{6378\,\text{km} + 27\,\text{km}}\right)^2$$

$$g_h = 9.724\,\text{m/s}^2$$

Ambient Temperature at 27 km: $T = 223.65\,\text{K}$

Terminal Speed (Mach 5): $v_1 = 1.499\,\text{km/s}$

$$v_1 = M\,\sqrt{k \cdot R \cdot T}$$

$$v_1 = 5\,\sqrt{1.4 \cdot 287\,\tfrac{\text{J}}{\text{kg}\cdot\text{K}} \cdot 223.65\,\text{K}}$$

$$v_1 = 1499\,\text{m/s}$$

Average Acceleration: $a = 3.0\,\text{m/s}^2$

Burn Time: $\Delta t = 500\,\text{s}$

$$\Delta t = \frac{\Delta v}{a}$$

$$\Delta t = \frac{v_1 - v_0}{a}$$

$$\Delta t = \frac{1499\,\text{m/s} - 0\,\text{m/s}}{3.0\,\text{m/s}^2}$$

$$\Delta t = 500\,\text{s}$$

Flight Path: $\Delta s = 374\,\text{km}$

$$\Delta s = \tfrac{1}{2}\, a \left(\Delta t\right)^2$$

$$\Delta s = \tfrac{1}{2}\, 3.0\,\text{m/s}^2 \left(500\,\text{s}\right)^2$$

$$\Delta s = 374\,\text{km}$$

Average Climb Angle: $\gamma = 4.1^\circ$

$$\gamma = \sin^{-1}(\Delta h / \Delta s)$$

$$\gamma = \sin^{-1}(27\,\text{km} / 374\,\text{km})$$

$$\gamma = 4.1^\circ$$

Average Lift-to-Drag Ratio: $L / D = 5.0$

Average Specific Impulse: $I_{sp} = 900\,\text{s}$

Burn Mass Ratio: $m_1 / m_0 = 0.726$

$$\frac{m_1}{m_0} = \exp\left[\frac{\Delta v + \left((1 - I/W)\cdot(L/D)^{-1} + \sin\gamma\right)\cdot g_h \cdot \Delta t}{-I_{sp}\cdot g_e}\right]$$

$$\frac{m_1}{m_0} = \exp\left[\frac{1499\,\text{m/s} + \left((1-0)\cdot(5.0)^{-1} + \sin 4.1^\circ\right)\cdot 9.724\,\text{m/s}^2 \cdot 500\,\text{s}}{-900\,\text{s}\cdot 9.807\,\text{m/s}^2}\right]$$

$$\frac{m_1}{m_0} = 0.726$$

Inertia-to-Weight Ratio at 27 km: $I/W = 0.036$

$$I/W = \frac{(v_1 \cdot \cos\gamma)^2}{(R_e + h)\, g_h}$$

$$I/W = \frac{(1499\,\text{m/s} \cdot \cos 4.1^\circ)^2}{(6378\,000\,\text{m} + 27\,000\,\text{m})\, 9.724\,\text{m/s}^2}$$

$$I/W = 0.036$$

Scramjet Burn: $27 < h < 51\,\text{km}$

Acceleration due to Gravity at 51 km: $g_h = 9.652\,\text{m/s}^2$

$$g_h = g_e \left(\frac{R_e}{R_e + h} \right)^2$$

$$g_h = 9.807\,\text{m/s}^2 \left(\frac{6378\,\text{km}}{6378\,\text{km} + 51\,\text{km}} \right)^2$$

$$g_h = 9.652\,\text{m/s}^2$$

Ambient Temperature at 51 km: $T = 270.65\,\text{K}$

Terminal Speed (Mach 20): $v_2 = 6.595\,\text{km/s}$

$$v_2 = M \sqrt{k \cdot R \cdot T}$$

$$v_2 = 20 \sqrt{1.4 \cdot 287 \tfrac{\text{J}}{\text{kg} \cdot \text{K}} \cdot 270.65\,\text{K}}$$

$$v_2 = 6595\,\text{m/s}$$

Average Acceleration: $a = 6.0\,\text{m/s}^2$

Burn Time: $\Delta t = 849\,\text{s}$

$$\Delta t = \frac{\Delta v}{a}$$

$$\Delta t = \frac{v_2 - v_1}{a}$$

$$\Delta t = \frac{6595\,\text{m/s} - 1499\,\text{m/s}}{6.0\,\text{m/s}^2}$$

$$\Delta t = 849\,\text{s}$$

Flight Path: $\Delta s = 3438\,\text{km}$

$$\Delta s = v_1\,\Delta t + \tfrac{1}{2}\,a\,\left(\Delta t\right)^2$$

$$\Delta s = 1499\,\text{m/s} \cdot 849\,\text{s} + \tfrac{1}{2}\,6.0\,\text{m/s}^2\,\left(849\,\text{s}\right)^2$$

$$\Delta s = 3438\,\text{km}$$

Average Climb Angle: $\gamma = 0.4^\circ$

$$\gamma = \sin^{-1}(\Delta h / \Delta s)$$

$$\gamma = \sin^{-1}(24\,\text{km} / 3438\,\text{km})$$

$$\gamma = 0.4^\circ$$

Average Lift-to-Drag Ratio: $L / D = 4.0$

Average Specific Impulse: $I_{sp} = 1800\,\text{s}$

Burn Mass Ratio: $m_2 / m_1 = 0.668$

$$\frac{m_2}{m_1} = \exp\left[\frac{\Delta v + \left((1 - I/W)\cdot(L/D)^{-1} + \sin\gamma\right)\cdot g_h \cdot \Delta t}{-I_{sp}\cdot g_e}\right]$$

$$\frac{m_2}{m_1} = \exp\left[\frac{6595\,\text{m/s} - 1499\,\text{m/s} + \left((1 - 0.036)\cdot(4.0)^{-1} + \sin 0.4^\circ\right)\cdot 9.652\,\text{m/s}^2 \cdot 849\,\text{s}}{-1800\,\text{s}\cdot 9.807\,\text{m/s}^2}\right]$$

$$\frac{m_2}{m_1} = 0.668$$

Inertia-to-Weight Ratio at 51 km: $I / W = 0.701$

$$I / W = \frac{\left(v_2 \cdot \cos\gamma\right)^2}{\left(R_e + h\right) g_h}$$

$$I / W = \frac{\left(6595\,\text{m/s}\cdot\cos 0.4^\circ\right)^2}{\left(6378000\,\text{m} + 51000\,\text{m}\right) 9.652\,\text{m/s}^2}$$

$$I / W = 0.701$$

High Inertial Lift!

Using the Inertia-to-Weight Ratio at 27 km yields a greater fuel estimate.

Rocket Burn: $51 < h < 105\,\text{km}$

Acceleration due to Gravity at 105 km: $g_h = 9.492\,\text{m/s}^2$

$$g_h = g_e \left(\frac{R_e}{R_e + h} \right)^2$$

$$g_h = 9.807\,\text{m/s}^2 \left(\frac{6378\,\text{km}}{6378\,\text{km} + 105\,\text{km}} \right)^2$$

$$g_h = 9.492\,\text{m/s}^2$$

Ambient Temperature at 105 km: $T = 242.95\,\text{K}$

Terminal Speed (Circular Orbit): $v_3 = 7.841\,\text{km/s}$

$$v_3 = \sqrt{\frac{\mu}{R_e + h}}$$

$$v_3 = \sqrt{\frac{398600\,\text{km}^3/\text{s}^2}{6378\,\text{km} + 105\,\text{km}}}$$

$$v_3 = 7.841\,\text{km/s}$$

Terminal Mach Number: $M = 25.10$

$$M = \frac{v_3}{\sqrt{k \cdot R \cdot T}}$$

$$M = \frac{7841\,\text{m/s}}{\sqrt{1.4 \cdot 287\,\frac{\text{J}}{\text{kg}\cdot\text{K}} \cdot 242.95\,\text{K}}}$$

$$M = 25.10$$

Average Acceleration: $a = 9.0\,\text{m/s}^2$

Burn Time: $\Delta t = 138\,\text{s}$

$$\Delta t = \frac{\Delta v}{a}$$

$$\Delta t = \frac{v_3 - v_2}{a}$$

$$\Delta t = \frac{7841\,\text{m/s} - 6595\,\text{m/s}}{9.0\,\text{m/s}^2}$$

$$\Delta t = 138\,\text{s}$$

Flight Path: $\Delta s = 999\,\text{km}$

$$\Delta s = v_2\ \Delta t + \tfrac{1}{2}\, a\, (\Delta t)^2$$

$$\Delta s = 6595\,\text{m/s} \cdot 138\,\text{s} + \tfrac{1}{2}\, 9.0\,\text{m/s}^2\, (138\,\text{s})^2$$

$$\Delta s = 999\,\text{km}$$

Average Climb Angle: $\gamma = 3.1^\circ$

$$\gamma = \sin^{-1}(\Delta h / \Delta s)$$

$$\gamma = \sin^{-1}(54\,\text{km} / 999\,\text{km})$$

$$\gamma = 3.1^\circ$$

Average Lift-to-Drag Ratio: $L / D = 3.0$

Average Specific Impulse: $I_{sp} = 450\,\text{s}$

Burn Mass Ratio: $m_3 / m_2 = 0.720$

$$\frac{m_3}{m_2} = \exp\left[\frac{\Delta v + \left((1 - I/W) \cdot (L/D)^{-1} + \sin\gamma\right) \cdot g_h \cdot \Delta t}{-I_{sp} \cdot g_e}\right]$$

$$\frac{m_3}{m_3} = \exp\left[\frac{7841\,\text{m/s} - 6595\,\text{m/s} + \left((1 - 0.701) \cdot (3.0)^{-1} + \sin 3.1^\circ\right) \cdot 9.492\,\text{m/s}^2 \cdot 138\,\text{s}}{-450\,\text{s} \cdot 9.807\,\text{m/s}^2}\right]$$

$$\frac{m_3}{m_2} = 0.720$$

Inertia-to-Weight Ratio at 105 km: $I / W = 0.996$

$$I / W = \frac{(v_3 \cdot \cos\gamma)^2}{(R_e + h)\, g_h}$$

$$I / W = \frac{(7841\,\text{m/s} \cdot \cos 3.1^\circ)^2}{(6378000\,\text{m} + 105000\,\text{m})\, 9.492\,\text{m/s}^2}$$

$$I / W = 0.996$$

High Inertial Lift!

Using the Inertia-to-Weight Ratio at 51 km yields a greater fuel estimate.

Ascent Mass Ratio: $0 < h < 105\,\text{km}$

$$\frac{m_3}{m_0} = \frac{m_3}{m_2} \cdot \frac{m_2}{m_1} \cdot \frac{m_1}{m_0}$$

$$\frac{m_3}{m_0} = 0.720 \cdot 0.668 \cdot 0.726$$

$$\frac{m_3}{m_0} = 0.349$$

Fuel Mass Fraction: Fuel = $LH_2 + LO_2$

$$\frac{m_f}{m_0} = 1 - \frac{m_3}{m_0}$$

$$\frac{m_f}{m_0} = 1 - 0.349$$

$$\frac{m_f}{m_0} = 0.651$$

Vehicle Takeoff Mass: $m_0 = 250\,000\,\text{kg}$

Ascent Fuel Mass: $m_f = 162\,654\,\text{kg}$

$$m_f = \frac{m_f}{m_0} m_0$$

$$m_f = 0.651 \cdot 250\,000\,\text{kg}$$

$$m_f = 162\,654\,\text{kg}$$

Ascent Path: $s_{asc} = 4811\,\text{km}$

$$s_{asc} = \sum \Delta s$$

$$s_{asc} = 374\,\text{km} + 3438\,\text{km} + 999\,\text{km}$$

$$s_{asc} = 4811\,\text{km}$$

Ascent Time: $t_{asc} = 1487\,\text{s}$

$$t_{asc} = \sum \Delta t$$

$$t_{asc} = 500\,\text{s} + 849\,\text{s} + 138\,\text{s}$$

$$t_{asc} = 1487\,\text{s}$$